Problem 1

Write the program for numerical solution of the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{1}$$

for the diffusion equation on the interval $[0, 1]$. The input data for the program are: a function $u_0 : [0, 1] \to \mathbb{R}$ and a diffusion coefficient $D > 0$.

(a) Assume $D = 1$. Check the results of the program for two initial functions: (i) $u_0(x) = 1$ and (ii) $u_0(x) = \sin 2\pi x$. In the second case there is a simple expression for the exact solution $u_{exact}(x) = e^{-4\pi^2 t} \sin 2\pi x$, which you can use for comparison.

(b) Test the behavior of the results for longer times (e.g. $T \geq 10$) for the above initial functions, when different grid sizes are taken. Especially observe what happens when the stability condition $k \leq 0.5Dh^2$ is not met (Here k is time step and h is space step).

Problem 2

Perform the same exercises as in the **Problem 1**, but this time use the implicit scheme for the numerical solution of the diffusion equation. Pay special attention to the stability behavior and notice that if the condition $k \leq 0.5Dh^2$ is not met, the scheme does not produce growing values for longer time levels.ar interpolation between the boundary

Problem 3

Expand the numerical treatment of diffusion equation problem so that non-zero boundary conditions could be handled. Specifically consider the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = d_0(t), & \\ u(1,t) = d_1(t), & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases} \tag{2}$$

where $d_0(t)$ and $d_1(t)$ are given functions of time. Implement both the explicit and implicit schemes. Test the program for the following conditions:

$$d_0(t) = 1, \; d_1(t) = 3, \quad u_0(x) = 3x(1 - x).$$

The exact solution for long time approaches the linear interpolation between, that is

$$\lim_{t \to \infty} u(x,t) = 2x + 1. \tag{3}$$

Check whether this convergence is maintained by the approximate solutions. To do this carry out the calculations for $T = 5, 10, 20$ and compare results with (3).